4.3 Derivatives of Inverse Trigonometric Functions

What you will learn about . . .

- Derivatives of Inverse Functions
- Derivative of the Arcsine
- Derivative of the Arctangent
- Derivative of the Arcsecant
- Derivatives of the Other Three

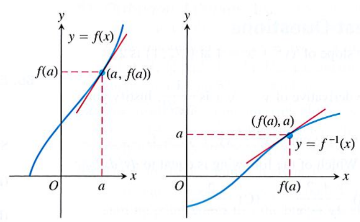
and why . . .

The relationship between the graph of a function and its inverse allows us to see the relationship between their derivatives.

Derivatives of Inverse Functions

In Section 1.5 we learned that the graph of the inverse of a function f can be obtained by reflecting the graph of f across the line $y = x$. If we combine that with our understanding of what makes a function differentiable, we can gain some quick insights into the differentiability of inverse functions.

As Figure 4.12 suggests, the reflection of a continuous curve with no cusps or corners will be another continuous curve with no cusps or corners. Indeed, if there is a tangent line to the graph of f at the point $(a, f(a))$, then that line will reflect across $y = x$ to become a tangent line to the graph of f^{-1} at the point $(f(a), a)$. We can even see geometrically that the *slope* of the reflected tangent line (when it exists and is not zero) will be the *reciprocal* of the slope of the original tangent line, since a change in y becomes a change in x in the reflection, and a change in x becomes a change in y.



The slopes are reciprocal: $\left.\dfrac{df^{-1}}{dx}\right|_{f(a)} = \dfrac{1}{\left.\dfrac{df}{dx}\right|_{a}}$

Figure 4.12 The graphs of a function and its inverse. Notice that the tangent lines have reciprocal slopes.

All of this serves as an introduction to the following theorem, which we will assume as we proceed to find derivatives of inverse functions. Although the essentials of the proof are illustrated in the geometry of Figure 4.12, a careful analytic proof is more appropriate for an advanced calculus text and will be omitted here.

THEOREM 1 Derivatives of Inverse Functions

If f is differentiable at every point of an interval I and df/dx is never zero on I, then f has an inverse and f^{-1} is differentiable at every point of the interval $f(I)$.

EXPLORATION 1 Finding a Derivative on an Inverse Graph Geometrically

Let $f(x) = x^5 + 2x - 1$. Since the point $(1, 2)$ is on the graph of f, it follows that the point $(2, 1)$ is on the graph of f^{-1}. Can you find

$$\frac{df^{-1}}{dx}(2),$$

the value of df^{-1}/dx at 2, without knowing a formula for f^{-1}?

1. Graph $f(x) = x^5 + 2x - 1$. A function must be one-to-one to have an inverse function. Is this function one-to-one?
2. Find $f'(x)$. How could this derivative help you to conclude that f has an inverse?
3. Reflect the graph of f across the line $y = x$ to obtain a graph of f^{-1}.
4. Sketch the tangent line to the graph of f^{-1} at the point $(2, 1)$. Call it L.
5. Reflect the line L across the line $y = x$. At what point is the reflection of L tangent to the graph of f?
6. What is the slope of the reflection of L?
7. What is the slope of L?
8. What is $\dfrac{df^{-1}}{dx}(2)$?

Derivative of the Arcsine

We know that the function $x = \sin y$ is differentiable in the open interval $-\pi/2 < y < \pi/2$ and that its derivative, the cosine, is positive there. Theorem 1 therefore assures us that the inverse function $y = \sin^{-1}(x)$ (the *arcsine* of x) is differentiable throughout the interval $-1 < x < 1$. We cannot expect it to be differentiable at $x = -1$ or $x = 1$, however, because the tangents to the graph are vertical at these points (Figure 4.13).

We find the derivative of $y = \sin^{-1}(x)$ as follows:

$$y = \sin^{-1} x$$

$$\sin y = x \qquad \text{Inverse function relationship}$$

$$\frac{d}{dx}(\sin y) = \frac{d}{dx}x \qquad \text{Differentiate both sides.}$$

$$\cos y\frac{dy}{dx} = 1 \qquad \text{Implicit differentiation}$$

$$\frac{dy}{dx} = \frac{1}{\cos y}$$

The division in the last step is safe because $\cos y \neq 0$ for $-\pi/2 < y < \pi/2$. In fact, $\cos y$ is *positive* for $-\pi/2 < y < \pi/2$, so we can replace $\cos y$ with $\sqrt{1 - (\sin y)^2}$, which is $\sqrt{1 - x^2}$. Thus

$$\frac{d}{dx}(\sin^{-1} x) = \frac{1}{\sqrt{1 - x^2}}.$$

If u is a differentiable function of x with $|u| < 1$, we apply the Chain Rule to get

$$\frac{d}{dx}\sin^{-1} u = \frac{1}{\sqrt{1 - u^2}}\frac{du}{dx}, \quad |u| < 1.$$

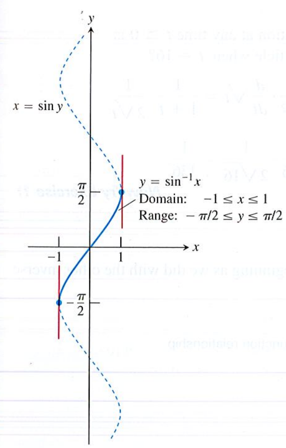


Figure 4.13 The graph of $y = \sin^{-1} x$ has vertical tangents $x = -1$ and $x = 1$.

EXAMPLE 1 Applying the Formula

$$\frac{d}{dx}(\sin^{-1} x^2) = \frac{1}{\sqrt{1-(x^2)^2}} \cdot \frac{d}{dx}(x^2) = \frac{2x}{\sqrt{1-x^4}}$$

Now Try Exercise 3.

Derivative of the Arctangent

Although the function $y = \sin^{-1}(x)$ has a rather narrow domain of $[-1, 1]$, the function $y = \tan^{-1} x$ is defined for all real numbers, and is differentiable for all real numbers, as we will now see. The differentiation proceeds exactly as with the arcsine function.

$$y = \tan^{-1} x$$

$$\tan y = x \qquad \text{Inverse function relationship}$$

$$\frac{d}{dx}(\tan y) = \frac{d}{dx}x$$

$$\sec^2 y \frac{dy}{dx} = 1 \qquad \text{Implicit differentiation}$$

$$\frac{dy}{dx} = \frac{1}{\sec^2 y}$$

$$= \frac{1}{1+(\tan y)^2} \qquad \text{Trig identity: } \sec^2 y = 1 + \tan^2 y$$

$$= \frac{1}{1+x^2}$$

The derivative is defined for all real numbers. If u is a differentiable function of x, we get the Chain Rule form:

$$\frac{d}{dx}\tan^{-1} u = \frac{1}{1+u^2}\frac{du}{dx}.$$

EXAMPLE 2 A Moving Particle

A particle moves along the x-axis so that its position at any time $t \geq 0$ is $x(t) = \tan^{-1}\sqrt{t}$. What is the velocity of the particle when $t = 16$?

SOLUTION $$v(t) = \frac{d}{dt}\tan^{-1}\sqrt{t} = \frac{1}{1+(\sqrt{t})^2} \cdot \frac{d}{dt}\sqrt{t} = \frac{1}{1+t} \cdot \frac{1}{2\sqrt{t}}$$

When $t = 16$, the velocity is $v(16) = \dfrac{1}{1+16} \cdot \dfrac{1}{2\sqrt{16}} = \dfrac{1}{136}$.

Now Try Exercise 11.

Derivative of the Arcsecant

We find the derivative of $y = \sec^{-1} x$, $|x| > 1$, beginning as we did with the other inverse trigonometric functions.

$$y = \sec^{-1} x$$

$$\sec y = x \qquad \text{Inverse function relationship}$$

$$\frac{d}{dx}(\sec y) = \frac{d}{dx}x$$

$$\sec y \tan y \frac{dy}{dx} = 1$$

$$\frac{dy}{dx} = \frac{1}{\sec y \tan y} \qquad \text{Since } |x| > 1, y \text{ lies in } (0, \pi/2) \cup (\pi/2, \pi) \text{ and } \sec y \tan y \neq 0.$$

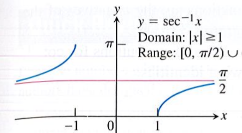


Figure 4.14 The slope of the curve $y = \sec^{-1} x$ is positive for both $x < -1$ and $x > 1$.

To express the result in terms of x, we use the relationships

$$\sec y = x \quad \text{and} \quad \tan y = \pm\sqrt{\sec^2 y - 1} = \pm\sqrt{x^2 - 1}$$

to get

$$\frac{dy}{dx} = \pm\frac{1}{x\sqrt{x^2-1}}.$$

Can we do anything about the $\pm$ sign? A glance at Figure 4.14 shows that the slope of the graph $y = \sec^{-1} x$ is always positive. That must mean that

$$\frac{d}{dx}\sec^{-1} x = \begin{cases} +\dfrac{1}{x\sqrt{x^2-1}} & \text{if } x > 1 \\ -\dfrac{1}{x\sqrt{x^2-1}} & \text{if } x < -1. \end{cases}$$

With the absolute value symbol we can write a single expression that eliminates the "$\pm$" ambiguity:

$$\frac{d}{dx}\sec^{-1} x = \frac{1}{|x|\sqrt{x^2-1}}.$$

If u is a differentiable function of x with $|u| > 1$, we have the formula

$$\frac{d}{dx}\sec^{-1} u = \frac{1}{|u|\sqrt{u^2-1}}\frac{du}{dx}, \quad |u| > 1.$$

EXAMPLE 3 Using the Formula

$$\frac{d}{dx}\sec^{-1}(5x^4) = \frac{1}{|5x^4|\sqrt{(5x^4)^2-1}}\frac{d}{dx}(5x^4)$$

$$= \frac{1}{5x^4\sqrt{25x^8-1}}(20x^3)$$

$$= \frac{4}{x\sqrt{25x^8-1}}$$

Now Try Exercise 17.

Derivatives of the Other Three

We could use the same technique to find the derivatives of the other three inverse trigonometric functions—arccosine, arccotangent, and arccosecant—but there is a much easier way, thanks to the following identities.

Inverse Function–Inverse Cofunction Identities

$$\cos^{-1} x = \pi/2 - \sin^{-1} x$$
$$\cot^{-1} x = \pi/2 - \tan^{-1} x$$
$$\csc^{-1} x = \pi/2 - \sec^{-1} x$$

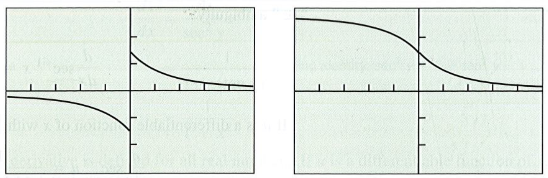
It follows easily that the derivatives of the inverse cofunctions are the negatives of the derivatives of the corresponding inverse functions (see Exercises 32–34).

You have probably noticed by now that most calculators do not have buttons for $\cot^{-1}$, $\sec^{-1}$, or $\csc^{-1}$. They are not needed because of the following identities:

Calculator Conversion Identities

$$\sec^{-1} x = \cos^{-1}(1/x)$$
$$\cot^{-1} x = \pi/2 - \tan^{-1} x$$
$$\csc^{-1} x = \sin^{-1}(1/x)$$

Notice that we do not use $\tan^{-1}(1/x)$ as an identity for $\cot^{-1} x$. A glance at the graphs of $y = \tan^{-1}(1/x)$ and $y = \pi/2 - \tan^{-1}x$ reveals the problem (Figure 4.15).



[–5, 5] by [–3, 3]
(a)

[–5, 5] by [–3, 3]
(b)

Figure 4.15 The graphs of (a) $y = \tan^{-1}(1/x)$ and (b) $y = \pi/2 - \tan^{-1} x$. The graph in (b) is the same as the graph of $y = \cot^{-1} x$.

We cannot replace $\cot^{-1} x$ by the function $y = \tan^{-1}(1/x)$ in the identity for the inverse functions and inverse cofunctions, and so it is not the function we want for $\cot^{-1} x$. The ranges of the inverse trigonometric functions have been chosen in part to make the two sets of identities above hold.

EXAMPLE 4 A Tangent Line to the Arccotangent Curve

Find an equation for the line tangent to the graph of $y = \cot^{-1} x$ at $x = -1$.

SOLUTION

First, we note that

$$\cot^{-1}(-1) = \pi/2 - \tan^{-1}(-1) = \pi/2 - (-\pi/4) = 3\pi/4.$$

The slope of the tangent line is

$$\left.\frac{dy}{dx}\right|_{x=-1} = -\left.\frac{1}{1+x^2}\right|_{x=-1} = -\frac{1}{1+(-1)^2} = -\frac{1}{2}.$$

So the tangent line has equation $y - 3\pi/4 = (-1/2)(x + 1)$.

Now Try Exercise 23.

Quick Review 4.3 *(For help, go to Sections 1.2, 1.5, and 1.6.)*

Exercise numbers with a gray background indicate problems that the authors have designed to be solved *without a calculator.*

In Exercises 1–5, give the *domain* and *range* of the function, and evaluate the function at $x = 1$.

1. $y = \sin^{-1} x$
2. $y = \cos^{-1} x$
3. $y = \tan^{-1} x$;
4. $y = \sec^{-1} x$
5. $y = \tan(\tan^{-1} x)$

In Exercises 6–10, find the inverse of the given function.

6. $y = 3x - 8$
7. $y = \sqrt[3]{x + 5}$
8. $y = \dfrac{8}{x}$
9. $y = \dfrac{3x - 2}{x}$
10. $y = \arctan(x/3)$

Section 4.3 Exercises

In Exercises 1–8, find the derivative of y with respect to the appropriate variable.

1. $y = \cos^{-1}(x^2)$
2. $y = \cos^{-1}(1/x)$
3. $y = \sin^{-1}\sqrt{2}t$
4. $y = \sin^{-1}(1 - t)$
5. $y = \sin^{-1}\dfrac{3}{t^2}$
6. $y = s\sqrt{1 - s^2} + \cos^{-1} s$
7. $y = x\sin^{-1}x + \sqrt{1 - x^2}$
8. $y = \dfrac{1}{\sin^{-1}(2x)}$

In Exercises 9–12, a particle moves along the x-axis so that its position at any time $t \ge 0$ is given by $x(t)$. Find the velocity at the indicated value of t.

9. $x(t) = \sin^{-1}\left(\dfrac{t}{4}\right)$, $t = 3$
10. $x(t) = \sin^{-1}\left(\dfrac{\sqrt{t}}{4}\right)$, $t = 4$
11. $x(t) = \tan^{-1}t$, $t = 2$
12. $x(t) = \tan^{-1}(t^2)$, $t = 1$

In Exercises 13–22, find the derivatives of y with respect to the appropriate variable.

13. $y = \sec^{-1}(2s + 1)$
14. $y = \sec^{-1} 5s$
15. $y = \csc^{-1}(x^2 + 1)$, $x > 0$
16. $y = \csc^{-1} x/2$
17. $y = \sec^{-1}\dfrac{1}{t}$, $0 < t < 1$
18. $y = \cot^{-1}\sqrt{t}$
19. $y = \cot^{-1}\sqrt{t - 1}$
20. $y = \sqrt{s^2 - 1} - \sec^{-1} s$
21. $y = \tan^{-1}\sqrt{x^2 - 1} + \csc^{-1} x$, $x > 1$
22. $y = \cot^{-1}\dfrac{1}{x} - \tan^{-1} x$

In Exercises 23–26, find an equation for the tangent to the graph of y at the indicated point.

23. $y = \sec^{-1} x$, $x = 2$
24. $y = \tan^{-1} x$, $x = 2$
25. $y = \sin^{-1}\left(\dfrac{x}{4}\right)$, $x = 3$
26. $y = \tan^{-1}(x^2)$, $x = 1$

27. (a) Find an equation for the line tangent to the graph of $y = \tan x$ at the point $(\pi/4, 1)$.
    (b) Find an equation for the line tangent to the graph of $y = \tan^{-1} x$ at the point $(1, \pi/4)$.
28. Let $f(x) = x^5 + 2x^3 + x - 1$.
    (a) Find $f(1)$ and $f'(1)$.
    (b) Find $f^{-1}(3)$ and $(f^{-1})'(3)$.
29. Let $f(x) = \cos x + 3x$.
    (a) Show that f has a differentiable inverse.
    (b) Find $f(0)$ and $f'(0)$.
    (c) Find $f^{-1}(1)$ and $(f^{-1})'(1)$.
30. **Group Activity** Graph the function $f(x) = \sin^{-1}(\sin x)$ in the viewing window $[-2\pi, 2\pi]$ by $[-4, 4]$. Then answer the following questions:
    (a) What is the domain of f?
    (b) What is the range of f?
    (c) At which points is f not differentiable?
    (d) Sketch a graph of $y = f'(x)$ without using NDER or computing the derivative.
    (e) Find $f'(x)$ algebraically. Can you reconcile your answer with the graph in part (d)?
31. **Group Activity** A particle moves along the x-axis so that its position at any time $t \ge 0$ is given by $x = \arctan t$.
    (a) Prove that the particle is always moving to the right.
    (b) Prove that the particle is always decelerating.
    (c) What is the limiting position of the particle as t approaches infinity?

In Exercises 32–34, use the inverse function–inverse cofunction identities to derive the formula for the derivative of the function.

32. arccosine
33. arccotangent
34. arccosecant

Standardized Test Questions

You may use a graphing calculator to solve the following problems.

35. True or False The domain of $y = \sin^{-1} x$ is $-1 \le x \le 1$. Justify your answer.

36. True or False The domain of $y = \tan^{-1} x$ is $-1 \le x \le 1$. Justify your answer.

37. Multiple Choice Which of the following is $\dfrac{d}{dx}\sin^{-1}\left(\dfrac{x}{2}\right)$?

(A) $-\dfrac{2}{\sqrt{4-x^2}}$ (B) $-\dfrac{1}{\sqrt{4-x^2}}$ (C) $\dfrac{2}{4+x^2}$

(D) $\dfrac{2}{\sqrt{4-x^2}}$ (E) $\dfrac{1}{\sqrt{4-x^2}}$

38. Multiple Choice Which of the following is $\dfrac{d}{dx}\tan^{-1}(3x)$?

(A) $-\dfrac{3}{1+9x^2}$ (B) $-\dfrac{1}{1+9x^2}$ (C) $\dfrac{1}{1+9x^2}$

(D) $\dfrac{3}{1+9x^2}$ (E) $\dfrac{3}{\sqrt{1-9x^2}}$

39. Multiple Choice Which of the following is $\dfrac{d}{dx}\sec^{-1}(x^2)$?

(A) $\dfrac{2}{x\sqrt{x^4-1}}$ (B) $\dfrac{2}{x\sqrt{x^2-1}}$ (C) $\dfrac{2}{x\sqrt{1-x^4}}$

(D) $\dfrac{2}{x\sqrt{1-x^2}}$ (E) $\dfrac{2x}{\sqrt{1-x^4}}$

40. Multiple Choice Which of the following is the slope of the tangent line to $y = \tan^{-1}(2x)$ at $x = 1$?

(A) $-2/5$ (B) $1/5$ (C) $2/5$ (D) $5/2$ (E) 5

Explorations

In Exercises 41–46, find **(a)** the right end behavior model, **(b)** the left end behavior model, and **(c)** any horizontal tangents for the function if they exist.

41. $y = \tan^{-1} x$ **42.** $y = \cot^{-1} x$

43. $y = \sec^{-1} x$ **44.** $y = \csc^{-1} x$

45. $y = \sin^{-1} x$ **46.** $y = \cos^{-1} x$

Extending the Ideas

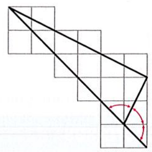
47. *Identities* Confirm the following identities for $x > 0$.

(a) $\cos^{-1} x + \sin^{-1} x = \pi/2$

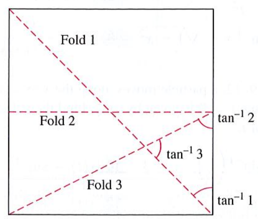
(b) $\tan^{-1} x + \cot^{-1} x = \pi/2$

(c) $\sec^{-1} x + \csc^{-1} x = \pi/2$

48. *Proof Without Words* The figure gives a proof without words that $\tan^{-1} 1 + \tan^{-1} 2 + \tan^{-1} 3 = \pi$. Explain what is going on.



49. *(Continuation of Exercise 48)* Here is a way to construct $\tan^{-1} 1$, $\tan^{-1} 2$, and $\tan^{-1} 3$ by folding a square of paper. Try it and explain what is going on.



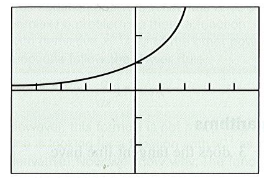
4.4 Derivatives of Exponential and Logarithmic Functions

What you will learn about . . .

- Derivative of e^x
- Derivative of a^x
- Derivative of $\ln x$
- Derivative of $\log_a x$
- Power Rule for Arbitrary Real Powers

and why . . .

Exponential functions are involved in the modeling of growth rates in the real world.



[–4.9, 4.9] by [–2.9, 2.9]

(a)

X	Y1
–.03	.98515
–.02	.99007
–.01	.99502
0	ERROR
.01	1.005
.02	1.0101
.03	1.0152

X=0

(b)

Figure 4.16 (a) The graph and (b) the table support the conclusion that

$$\lim_{h\to 0}\frac{e^h - 1}{h} = 1.$$

Derivative of e^x

At the end of the brief review of exponential functions in Section 1.3, we mentioned that the function $y = e^x$ was a particularly important function for modeling exponential growth. The number e was defined in that section to be the limit of $(1 + 1/x)^x$ as $x \to \infty$. This intriguing number shows up in other interesting limits as well, but the one with the most interesting implications for the *calculus* of exponential functions is this one:

$$\lim_{h\to 0}\frac{e^h - 1}{h} = 1.$$

(The graph and the table in Figure 4.16 provide strong support for this limit being 1. A formal algebraic proof that begins with our limit definition of e would require some rather subtle limit arguments, so we will not include one here.)

The fact that the limit is 1 creates a remarkable relationship between the function e^x and its derivative, as we will now see.

$$\begin{aligned}\frac{d}{dx}(e^x) &= \lim_{h\to 0}\frac{e^{x+h} - e^x}{h}\\ &= \lim_{h\to 0}\frac{e^x \cdot e^h - e^x}{h}\\ &= \lim_{h\to 0}\left(e^x \cdot \frac{e^h - 1}{h}\right)\\ &= e^x \cdot \lim_{h\to 0}\left(\frac{e^h - 1}{h}\right)\\ &= e^x \cdot 1\\ &= e^x\end{aligned}$$

In other words, the derivative of this particular function is itself!

$$\frac{d}{dx}(e^x) = e^x$$

If u is a differentiable function of x, then we have

$$\frac{d}{dx}e^u = e^u\frac{du}{dx}.$$

We will make extensive use of this formula when we study exponential growth and decay in Chapter 7.

EXAMPLE 1 Using the Formula

Find dy/dx if $y = e^{(x+x^2)}$.

SOLUTION

Let $u = x + x^2$ and $y = e^u$. Then

$$\frac{dy}{dx} = e^u\frac{du}{dx}, \quad \text{and} \quad \frac{du}{dx} = 1 + 2x.$$

Thus, $\dfrac{dy}{dx} = e^u\dfrac{du}{dx} = e^{(x+x^2)}(1 + 2x)$.

Now Try Exercise 9.